CHAPTER 5 MODIFICATION OF THRESHOLD TRANSFORMATIONS

Abstract. After the graph structures of self-dual one-to- one transformations are described, a construction method of generating minimal one-to-one threshold transformations from lower dimensional ones through face copies and orbit modifications are presented. Theorems that concern one-to-one threshold transformations and support the method are also proved. Then compression and expansion of threshold transformations are described. Then in addition to one-to-one threshold transformations in Chapter 4, some classes of incompressible and compressible one-to-one threshold transformations are given. In particular, all currently known one-to-one minimal incompressible threshold transformations are proved to be reflective. Finally, it is proved that there exists a threshold transformation of $\{0,1\}^n$ such that for any $1 \leq k \leq 2^n$, its graph has a k-cycle that is the only cycle or loop. This enhanced Arimoto theorem has an important implication on dynamical systems of neural networks described in the remaining chapters.

5.1 Orbit modification

If F is a threshold transformation, its graph drew considerable interest in the 1960s and early 70s (Arimoto, 1963; Masters & Mattson; 1966, Ishii, 1970; Ishii & Miyazaki, 1972; etc.). These studies provided some construction procedures for finding a threshold transformation that satisfies given conditions for its graph. However, they were mainly concerned with proper subgraphs of GRAPH(F) and did not bring out concrete results for the whole graph structure of a transformation except the ones given in Arimoto (1963) and Masters & Mattson (1966). Since what characterizes a transformation is its (whole) graph structure rather than the structure of its particular proper subgraph, our goal here is to fill part of that vacancy. In order to simplify the problem we limit our scope to graphs of one-to-one transformations and use the results of Chapters 3 and 4, which dealt with reflective transformations and circular one-to-one threshold transformations.

We have already proved that a one-to-one threshold transformation F is self-dual (Theorem 4.3.3 of Chapter 4), so that F can be represented by $F = [f_1, ..., f_n]$, where $f_i = p_i \cdot \neg(p_i F)$. Further, by Corollary 4.3.5 of Chapter 4, F is a threshold transformation, if and only if each f_i is a threshold function.

Theorem 2.3.5 of Chapter 2 completely determines the cycle structures of one-to-one self-dual transformations. However, such a condition for one-to-one threshold transformations is unknown. Therefore, it is necessary to try to construct a one-to-one threshold transformation individually in order to determine whether a given cycle structure is realized by a threshold transformation or not. In this section we prove two properties concerning face copies of one-to-one threshold transformations. Then. based on these results, we discuss a method of constructing higher dimensional one-to-one threshold transformations from lower dimensional ones. The method is orbit modification, which we applied in Chapter 3.5. It was part of more general construction procedures given by Ishii & Miyazaki (1972). Here we formulate it in terms of [ ]-representations.

Lemma 5.1.1 Let $M = \{n+1, ..., n+m\}$ and $C \subseteq \mathbf{Q}^M$ be a complete set. If g is a threshold function: $\mathbf{Q}^{n+m} \to \mathbf{Q}$ such that $g = f \cdot 1_C$ for some $f : \mathbf{Q}^{\mathbf{N}} \to \mathbf{Q}$ and the characteristic function $1_C : \mathbf{Q}^M \to \mathbf{Q}$, then $C = \mathbf{Q}^M$.

Proof. Let $g = f \cdot 1_C$ be a threshold function. Let $q \in g$; then $r = \{n+1, ..., n+m\}^- q \in g$. If $q\prime = (q_1, .., q_n, a_{n+1}, .., a_{n+m}) \notin g$ for some point $(a_{n+1}, ..., a_{n+m}) \in \mathbf{Q}^{\{n+1,...,n+m\}}$, then $r\prime = \{n+1, ..., n+m\}^- q\prime \notin g$. Since $q + r = q\prime + r\prime$, g is 2-summable, so that we have a contradiction to the fact that g is a threshold function (Corollary 4.2.2 of Chapter 4). □

Theorem 5.1.2 If G is a one-to-one threshold transformation of $\mathbf{Q}^{n+m}$, and none of the $(n+1)$th, ... and $(n+m)$th coordinates of any point of $\mathbf{Q}^{n+m}$ changes under G, then G is the $(n+1, ..., n+m)$th face copies of a one-to-one threshold transformation of $\mathbf{Q}^n$.

Proof. Let q be a point on a cycle of any transformation H of $\mathbf{Q}^{n+m}$. For any i, if q changes its ith coordinate from 1 to 0 under H, then another point on the same cycle must change its ith coordinate from 0 to 1 under H. Moreover, on the same cycle, the number of points that change their ith coordinates from 1 to 0 must be the same as the number of points that change their ith coordinates from 0 to 1. Let $M = \{n+1, ..., n+m\}$ and a be a point of $\mathbf{Q}^M$.

From the given condition, each cycle of G is contained either in $P_M^{-1}a$ or outside $P_M^{-1}a$, where P_M is the projection function: $\mathbf{Q}^{n+m} \to \mathbf{Q}^M$. Therefore, the number of $q \in P_M^{-1}a$ such that q changes its ith coordinate from 1 to 0 under G and the number of $r \in P_M^{-1}a$ such that r changes its ith coordinate from 0 to 1 under G are the same. Let $G = [g_1, .., g_n, \emptyset, .., \emptyset]$; then the number of $q \in g_i$ such that $P_M q = a$ and the number of $r \in g_i$ such that $P_M r = \bar{\neg} a$ are the same. Let q be an arbitrary point of g_i and $r = M^- q$. If $r \in \neg g_i$, then there exist $q\prime \in g_i$ and $r\prime \in \neg g_i$ such that $P_M q\prime = \neg P_M q$ and $r\prime = M^- q\prime$. Hence, $q + q\prime = r + r\prime$, q and $q\prime$ are points of g_i, and r and $r\prime$ are points of $\neg g_i$, so that g_i is 2-summable, contrary to the fact that g_i is a threshold function (Corollary 4.2.2 of Chapter 4). Hence, there exist some $f_i : \mathbf{Q}^{\mathbf{N}} \to \mathbf{Q}$ and a complete set $C_i \subseteq \mathbf{Q}^M$ such that $g_i = f_i \cdot 1_{Ci}$. Since $C_i = \mathbf{Q}^M$ by Lemma 5.1.1, we have $G = F \times I_{\mathbf{Q}^M}$, where $F = [f_1, ..., f_n]$, and F is a one-to-one threshold transformation. □

Theorem 5.1.3 Let G be a minimal one-to-one threshold transformation of $\mathbf{Q}^{n+1}$. Then there exists a self- dual threshold transformation F of $\mathbf{Q}^n$ such that if both $q \in \mathbf{Q}^{n+1}$ and $(n+1)^- q$ change or neither changes their ith coordinate under G for $i = 1, ..., n$, then $Gq = (F \times I_{\mathbf{Q}^{\{n+1\}}})q$.

Proof. Let $G = [g_1, ..., g_{n+1}]$. Let

$$f_i = (g_i|0) \cup (g_i|1) \text{ or } f_i = (g_i|0) \cap (g_i|1)$$

for $i = 1, ..., n$, where $0, 1 \in \mathbf{Q}^{\{n+1\}}$. Then $F = [f_1, ..., f_n]$ is a threshold transformation of $\mathbf{Q}^n$ by Corollary 4.2.4 of Chapter 4. If both a point $q \in \mathbf{Q}^{n+1}$ and $q\prime = (n+1)^- q$ change or neither changes their ith coordinate under G for $i = 1, ..., n$, and only one of them changes its $(n+1)$th coordinate, then $Gq = Gq\prime$, contrary to the fact that G is one-to- one. Suppose both q and $q\prime$ change their $(n+1)$th coordinates. Then $q \in g_{n+1}$ and $q\prime \in \bar{\neg} g_{n+1}$, or $q \in \bar{\neg} g_{n+1}$ and $q\prime \in g_{n+1}$. By Proposition 4.2.6 of Chapter 4, $|g_{n+1}| \geq 2^{n-1} + 1$; hence G is not minimal, because $\mathrm{Var}((n+1)^{-}G) < \mathrm{Var}(G)$. Therefore, neither q nor $q\prime$ changes its $(n+1)$th coordinate, and F satisfies the desired condition. □

Orbit modification Theorem 5.1.3 shows that any minimal one-to-one threshold transformation $G = [g_1, ..., g_{n+1}]$ of $\mathbf{Q}^{n+1}$ can be constructed from a self-dual threshold transformation $F = [f_1, ..., f_n]$ through its $(n+1)$th face copies $F \times I_{\mathbf{Q}^{\{n+1\}}} = [f_1 \circ P_{\mathbf{N}}, .., f_n \circ P_{\mathbf{N}}, \emptyset]$; g_i for $i = 1, ..., n$ is obtained by adding or deleting a subset of either the face $\neg p_{n+1}$ or p_{n+1} to or from $f_i \cdot 1_{\mathbf{Q}^{\{n+1\}}}$. Further, g_{n+1} is composed of q, $\bar{\neg} q$, $(n+1)^{-}q$ or $(n+1)^{-}\bar{\neg} q$ for some of these added or deleted points q. The $(n+1)$th coordinates of two of the four are 1, but g_{n+1} can not contain both because of the minimality of G. By Theorem 5.1.2, $g_{n+1} \neq \emptyset$ for G to be one-to-one, unless $G = F \times I_{\mathbf{Q}^{\{n+1\}}}$. The method of constructing G from $F \times I_{\mathbf{Q}^{\{n+1\}}}$ in this way is orbit modification described in 3.5. More generally, we may construct a new threshold transformation G from $F \times I_{\mathbf{Q}^{\{n+1,..,n+m\}}}$ by orbit modification.

5.2 Expansion and compression

let $D = \{D_1, ..., D_m\}$ be a partition of $\mathbf{N} = \mathbf{N}_n$, and let δ be a function from $\mathbf{N}_n$ to $\mathbf{M} = \mathbf{N}_m$ defined by $\delta j = i$ if $j \in D_i$. Let γ be a function from $\mathbf{M}$ to $\mathbf{N}$ such that γi is a representative element of D_i. Further let ϵ be a function from $\mathbf{N} \to \{I_{\mathbf{Q}}, \neg\}$ such that $\epsilon_j = I_{\mathbf{Q}}$ for every representative element $j \in \gamma\mathbf{M}$.

Let the function $H : \mathbf{Q}^m \to \mathbf{Q}^n$ be defined by

$$p_j H = \epsilon_j p_{\delta j} \quad \text{for every } j \in \mathbf{N}. \tag{5.2.1}$$

Clearly H is an injection. In particular, if $i \in \mathbf{M}$ and $j = \gamma i$, then $p_j H = p_{\delta j}$, since $\delta\gamma$ is the identity and $\epsilon_{\gamma i} = I_{\mathbf{Q}}$ for every $i \in \mathbf{M}$. Therefore,

$$p_i = p_{\gamma i} H for i \in \mathbf{M},$$

so that $p_{\delta j} = p_{\gamma\delta j} H$. Therefore, from (4.2.1),

$$p_j = (\epsilon j) p_{\gamma\delta j} \quad \text{for every } j \text{ on } H\mathbf{Q}^m. \tag{5.2.2}$$

Now, given a transformation F of $\mathbf{Q}^m$, define a transformation E of $\mathbf{Q}^n$ by

$$E = \begin{cases} HFH^{-1}, & \text{on } H\mathbf{Q}^m, \\ \text{Identity} & \text{on } (H\mathbf{Q}^m)^c. \end{cases} \tag{5.2.3}$$

E is caled an *expansion* of F.

Conversely given a transformation E of $\mathbf{Q}^n$, define $i \sim_F j$ for $i, j \in \mathbf{N}$ if $x_i = x_j$ for every $x \in \mathrm{Car}E \cup E(\mathrm{Car}E)$ or $x_i = \neg x_j$ for every $x \in \mathrm{Car}E \cup E(\mathrm{Car}E)$. Clearly $\sim_F$ is an equivalence relation on $\mathbf{N}$. Let $D = \{D_1, ..., D_m\}$ be the set of all equivalence classes defined by $\sim_F$. Let γi be a representative element of D_i. Let $\delta : \mathbf{N} \to \mathbf{M}$ be defined by $\delta j = i$ if $j \in D_i$. Then $p_j = \epsilon_j p_{\gamma\delta j}$ for every $j \in \mathbf{N}$ on $\mathrm{Car}E \cup E(\mathrm{Car}E)$, where $\epsilon_j = I_{\mathbf{Q}}$ (matrmidentity) or $\neg$. In particular, $\epsilon_{\gamma i} = I_{\mathbf{Q}}$ for every $i \in \mathbf{M}$. Therefore, the function $H : \mathbf{Q}^m \to \mathbf{Q}^n$ can be defined by (5.2.1). Define the transformation F of $\mathbf{Q}^m$ by

$$F = H^{-1} E H. \tag{5.2.4}$$

Then F is well-defined, since $\mathrm{Car}E \cup E(\mathrm{Car}E) \subseteq H\mathbf{Q}^m$. F is called a *compression* of E. Further, E is called *compressible*, if $|D_j| \geq 2$ for some j, and *incompressible* otherwise. F is self-dual if and only if E is self-dual.

Let $E = [e_1, ..., e_n]$, and let $F = [f_1, ..., f_m]$ be defined by (5.2.4). Then,

$$\begin{aligned} f_j &= p_j \cdot \neg(H^{-1}EH)_j \\ &= \epsilon_{\gamma j} p_{\delta\gamma j} \cdot \neg(EH)_{\gamma j} \\ &= p_{\gamma j}(\epsilon_1 p_{\delta 1}, ..., \epsilon_n p_{\delta n}) \cdot \neg E_{\gamma j}(\epsilon_1 p_{\delta 1}, ..., \epsilon_n p_{\delta n}) \\ &= e_{\gamma j}(\epsilon_1 p_{\delta 1}, ..., \epsilon_n p_{\delta n}). \end{aligned} \tag{5.2.5}$$

Conversely let E is defined by (5.2.3) from given F. Then, since (5.2.2) is necessary on $H\mathbf{Q}^m$,

$$e_i = p_i \cdot \neg E_i = p_i \cdot \neg(HFH^{-1})_i \cdot \Pi_{k\in\mathbf{N}}(p_k(=)\epsilon_k p_{\gamma\delta k}),$$

and

$$\begin{aligned} p_i \cdot \neg(HEH^{-1})_i &= \epsilon_i p_{\gamma\delta i} \cdot \neg\epsilon_i(EH^{-1})_{\delta i} \\ &= (\epsilon_i p_{\delta i} \cdot \neg\epsilon_i F_{\delta i})(p_{\gamma 1}, ..., p_{\gamma m}) \end{aligned}$$

Therefore,

$$e_i = \begin{cases} f_{\delta i}(p_{\gamma 1}, .., p_{\gamma m}) \cdot \Pi_{k\in\mathbf{N}}(p_k(=)\epsilon_k p_{\gamma\delta k}) & \text{if } \epsilon_i = I_{\mathbf{Q}}, \\ \bar{\neg} f_{\delta i}(p_{\gamma 1}, .., p_{\gamma m}) \cdot \Pi_{k\in\mathbf{N}}(p_k(=)\epsilon_k p_{\gamma\delta k}) & \text{if } \epsilon_i = \neg. \end{cases} \tag{5.2.6}$$

Example 5.2.1 Let $\mathbf{M} = \mathbf{N}_1$, $F = [p_1]$ be the transformation of $\mathbf{Q}^{\mathbf{M}}$, $\mathbf{N} = \mathbf{N}_n$, $D = \{\mathbf{N}\}$, $\gamma 1 = 1$, and $\epsilon_i = I_{\mathbf{Q}}$, for every $i \in \mathbf{M}$. $\mathrm{Car}E = \{1...1, 0...0\}$, and $Ex = \bar{\neg} x$. Therefore, $E = \langle e \rangle$,

$$e = p_1 \cdot p_2 \cdots p_n.$$

Proposition 5.2.2 Let $\mathbf{M} = \mathbf{N}_m$ and $\mathbf{N} = \mathbf{N}_{2m}$, $D = \{D_1, .., D_m\}$, $D_i = \{i, m+i\}$, and $\gamma i = i$, $\epsilon_{m+i} = \neg$ for every i. Let $F = \langle\langle f \rangle\rangle$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot ... \cdot \alpha_m p_m, \quad \text{where } \alpha_i = I_{\mathbf{Q}} \text{ or } \neg \text{ for every } i.$$

Then the expansion E of F is $\langle e \rangle$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot ... \cdot \alpha_m p_m \cdot \neg p_{m+1} \cdot \neg\alpha_2 p_{m+2} \cdot ... \cdot \neg\alpha_m p_{2m}. \tag{5.2.7}$$

Proof. Let ρ_m be the rotation $(1...m)$ and ρ_{2m} be the rotation $(1...2m)$. Let $f = \{x\}$, $p_1 x = 1$. Then, by (5.2.6),

$$\begin{aligned} e_i &= \{(\rho_m m-)^{i-1}x, \bar{\neg}(\rho_m m^{-})^{i-1}x)\} \\ &= \{(\neg x_{m+2-i}, .., \neg x_m, x_1, x_2, .., x_{m+2-(i+1)})\} \quad \text{for } i \in \mathbf{M}, \\ e_i &= \{(\bar{\neg}(\rho_m m^{-})^{-(i-m-1)}x, (\rho_m m^{-})^{-(i-m-1)}x)\} \\ &= \{(x_{2m+2-i}, .., x_m, \neg x_1, \neg x_2, .., \neg x_{2m+2-(i+1)})\} \quad \text{for } i \in \mathbf{N}\backslash\mathbf{M}. \\ \rho_{2m}^{i-1} e_1 &= \{(\rho_{2m}^{i-1}(x_1, .., x_m, \neg x_1, .., \neg x_m)\} \\ &= \{(\neg x_{m+2-i}, ..\neg x_m, x_1, .., x_{m-i+1})\} = e_i \quad \text{for } i \in \mathbf{M}, \\ \rho_{2m}^{i-1} e_1 &= \{(\rho_{2m}^{i-1}(x_1, .., x_m, \neg x_1, .., \neg x_m)\} \\ &= \{(x_{2m+2-i}, ..x_m, \neg x_1, .., \neg x_{2m-i+1})\} = e_i \quad \text{for } i \in \mathbf{N}\backslash\mathbf{M}. \end{aligned}$$

Therefore, $e_i = \rho_{2m}^{i-1} e_1$, that is, E is a circular ransformation $\langle e \rangle$ defined by (5.2.7). $\square$

Example 5.2.3 The simplest case in proposition 5.2.2 is when $f = p_1 \cdot .. \cdot p_m.$ Then $E = \langle e \rangle$,

$$e = p_1 \cdot p_2 \cdot ... \cdot p_m \cdot \neg p_{m+1} \cdot \neg p_{m+2} \cdot ... \cdot \neg p_{2m}.$$

The $2m$-cycle of E is

$$\begin{array}{ccccccc}
1...10...0 & \to & 01...10...0 & \to & ... & \to & 0...01...10 \\
\uparrow & & & & & & \downarrow \\
1...10...01 & \leftarrow & & & ... & \leftarrow & 0...01...1
\end{array}$$

E is uniquely minimal for $m \geq 2$.

Example 5.2.4 Let $0 < h - 1 < 2m$ be relatively prime with $2m$, and let $F = \langle\langle f \rangle\rangle$ be a one-to-one skew-circular transformation described in Theorem 4.5.3.

$$\begin{array}{rcl}
f & = & p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_m p_m, \\
F & = & (\rho m^-)^{h-1} \quad \text{on Car}F.
\end{array}$$

Then we obtain the expansion $E = \langle e \rangle$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_m p_m \cdot \neg p_{m+1} \cdot \neg \alpha_2 p_{m+2} \cdot .. \cdot \neg \alpha_m p_{2m}.$$

Thereore,

$$E = \rho^{h-1} \quad \text{on Car}E.$$

As an example, let

$$e = 1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot -5 \cdot -6 \cdot 7 \cdot \neg 8.$$

and compare the cycles of F and E.

$$\begin{array}{ccccccc}
1101 & \to & 0101 & \to & 0100 & \to & 0110 \\
\uparrow & & & & & & \downarrow \\
1001 & \leftarrow & 1011 & \leftarrow & 1010 & \leftarrow & 0010 \\
11010010 & \to & 01011010 & \to & 01001011 & \to & 01101001 \\
\uparrow & & & & & & \downarrow \\
10010110 & \leftarrow & 10110100 & \leftarrow & 10100101 & \leftarrow & 00101101
\end{array}$$

We have also the following proposition similar to Proposition 5.2.2.

Propositin 5.2.5 Let $\mathbf{N} = \mathbf{N}_{2m}$, $D = \{D_1, ..., D_m\}$, $D_i = \{i, m+i\}$, $\gamma i = i$, and $\epsilon(m+i) = I_{\mathbf{Q}}$ for every i. Let $F = \langle f \rangle$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot ... \cdot \alpha_m p_m, \quad \text{where } \alpha_i = I_{\mathbf{Q}} \text{ or } \neg \text{ for every } i.$$

Then the expamsion E of F is a circular transformation $\langle e \rangle$,

$$e = p_1 \cdot \epsilon_2 p_2 \cdot ... \cdot \epsilon_m p_m \cdot p_{m+1} \cdot \epsilon_2 p_{n+2} \cdot ... \cdot \epsilon_m p_{2m}.$$

Example 5.2.6 Let $0 < h - 1 < m$ be relatively prime with odd m, and let $F = \langle f \rangle$ be a one-to-one skew-circular transformation described in Theorem 4.4.3.

$$\begin{array}{rcl}
f & = & p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_m p_m, \\
F & = & \bar{\neg} \rho^{h-1} \quad \text{on Car}F.
\end{array}$$

Then we obtain the expansion $E = \langle e \rangle$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_m p_m \cdot p_{m+1} \cdot \alpha_2 p_{m+2} \cdot .. \cdot \alpha_m p_{2m},$$

Therefore,

$$E = \bar{\neg} \rho^{h-1} \quad \text{on Car}E.$$

In general, if F is a threshold transformation, then its compression E is also a threshold transformation, but the converse is not necessarily true. However, we have the following theorem.

Proposition 5.2.7 F is a self-dual threshold transformation if and only if, for its compression $E = [e_1, e_2, .., e_m]$ defined by (5.2.2), e_j is a threshold function for every j, and for each l such that $|\mathbf{M}_t| \geq 2$ and for each j,

$$e_j \subseteq p_t \quad \text{or} \quad e_j \subseteq \neg p_t.$$

Proof. Let $F = [f_1, ..., f_n]$ be a self-dual transformation of $\mathbf{Q}^n$ and $E = [e_1, ..., e_m]$ be a self-dual transformation of $\mathbf{Q}^m$ defined by (5.2.2). Suppose that there exist $y, y\prime \in e_j$ such that $y_t = 0$ and $y\prime_t = 1$ for some t such that $|\mathbf{M}_t| \geq 2$. Then there exists some s such that $s \neq \gamma t$ and $t = \delta s$. Let $i = \gamma j$, $Hy = x$, and $Hy\prime = x\prime$. Then $\delta i = j$. Also, $x, x\prime \in e_{\delta i}(p_{\gamma 1}, .., p_{\gamma m})$, $x_k = \epsilon_k x_{\gamma \delta k}$, and $x\prime_k = \epsilon_k x\prime_{\gamma \delta k}$ for every k, so that $x, x\prime \in f_i$ by (5.2.3). Further,

$$x_{\gamma t} = y_t = 0,\ x\prime_{\gamma t} = 1,\ x_s = \epsilon_s y_{\delta s} = \epsilon_s y_t = \epsilon_s 0,\ x\prime_s = \epsilon_s 1.$$

Let z and $z\prime$ of $\mathbf{Q}^n$ be defined as

$$\begin{aligned} &z_{\gamma t} = 0, \quad z_s = \epsilon_s 1, \quad z_u = x_u \text{ for every other } u; \\ &z\prime_{\gamma t} = 1, \quad z\prime_s = \epsilon_s 0, \quad z\prime_u = x\prime_u \text{ for every other } u. \end{aligned}$$

Since $\gamma t \sim_F s$, both z and $z\prime$ are fixed points of F, so that they are elements of $\neg f_i$. However, $x + x\prime = z + z\prime$ in R^n, i.e. f_i is 2-summable, contrary to the fact that f_i is a threshold function. The "if" part of this theorem is obvious. □

As seen from Proposition 5.2.7, an expansion of an incompressible threshold transformation may be a threshold transformation or not. If there exists some threshold transformation which is an expansion of a given threshold transformation, then the given threshold transformation may be called expansible. Thus, threshold transformations can be classified as follows.

$$\begin{array}{ccc} & Threshold transformations & \\ & \swarrow \quad \searrow & \\ Compressible & & Incompressible \end{array}$$

$$\begin{array}{ccc} & Threshold transformations & \\ & \swarrow \quad \searrow & \\ Expansible & & Inexpansible \end{array}$$

5.3 Incompressible transformations

We gave circular and skew-circular threshold transformations in Sections 4.4 and 4.5. In this section we try to construct general, minimal, non-compressible, one-to-one threshold transformations by orbit modification discussed in Section 5.1, although I have found only one example except trivial ones, which are face copies of or isometrically similar to circular or skew- circular transformations.

In the end of Chapter 4, we raised an open question about whether all minimal one-to-one threshold transformations are reflective through some isometries of order 2, so that their inverses are also threshold transformations. We have not solved this question, but we will show that all currently known one-to-one threshold transformations that are minimal and incompressible are reflective through some isometries of order 2. Combined with the result in Chapter 3.2 that any isometry is reflective through some isometry of order 2, these results tentatively indicate that one-to-one threshold transformations do not completely digress from isometries, despite selectiveness for non-fixed points in the former. These results also help to find a one-to-one threshold transformation.

Constructing a threshold bijection is in turn a good preparation for constructing a neural network, since not only similar methods can be applied to the two subjects, but also a dynamical neural network having an attractor is often constructed by modifying a one-to-one threshold transformation, as shown in Chapters 6 and 7. In the following, the expression of cycle structures introduced in Chapter 2.3 is used.

Example 5.3.1 Let $n = 2m + 2$. Consider the transformation F of Example 5.3.2 (3) for dimension $n - 1$, i.e., $F = [f, \rho f, ..., \rho^{n-2} f]$, where

$$\begin{aligned} f &= f_1 \vee f_2, \\ f_1 &= p_1 \cdots p_{m+1} \cdot \neg p_{m+2} \cdots \neg p_{2m+1}, \\ f_2 &= p_1 \cdots p_m \cdot \neg p_{m+1} \cdots \neg p_{2m+1}. \end{aligned}$$

The non-fixed points of the nth face copies of F form four $(n-1)$-cycles, for example,

$$\begin{array}{l} 111001 \to 011101 \to 001111 \to 100111 \to 110011 \to 111001, \\ 110001 \to 011001 \to 001101 \to 000111 \to 100011 \to 110001, \\ 111000 \to 011100 \to 001110 \to 100110 \to 110010 \to 111000, \\ 110000 \to 011000 \to 001100 \to 000110 \to 100010 \to 110000. \end{array}$$

Since F is reflective through the linear permutation ν of coefficients $(-1, n)$, i.e. $\nu = (1, n-1)(2, n-2) \cdots (n/2 - 1, n/2 + 1)$, its nth face copies are also reflective through ν by Proposition 3.5.1 (i) of Chapter 3. Let c be the point defined by $c_i = 1$ for $1 \le i \le n/2 - 1$, $c_i = 0$ for $n/2 \le i \le n - 1$, and $c_n = 1$, and let d be the point defined by $d_i = 1$ for $1 \le i \le n/2$, and $d_i = 0$ for $n/2 + 1 \le i \le n$ ($c = 110001$, $d = 111000$ in this example). Then applying Proposition 3.5.4 of Chapter 3, we obtain

$$\begin{aligned} G &= [g_1, ..., g_n], \\ g_i &= \rho^{i-1} f_1 \vee \rho^{i-1} f_2 \cdot \neg p_n \text{ for } i = 1, ..., m+1, \\ g_i &= \rho^{i-1} f_1 \cdot p_n \vee \rho^{i-1} f_2 \text{ for } i = m+2, ..., 2m+1, \\ g_n &= f_2 \cdot p_n. \end{aligned}$$

G is a threshold transformation, since g_i is a threshold function for every i. F is reflective through ν, incompressible, inexapansible, and

$$\mathrm{CS}(G) = \{(1, n), (2, n-1), (2^n - 3n + 2, 1)\}.$$

In the above example, the four cycles are changed into

$$\begin{array}{ccccccccccc} 111001 & \to & 011101 & \to & 001111 & \to & 100111 & \to & 110011 & \to & 111001, \\ 110001 & & & & & & 000111 & \to & 100011 & \to & 110001, \\ \downarrow & & & & & \nearrow & & & & & \\ 111000 & \to & 011100 & \to & 001110 & & & & & & \\ 110000 & \to & 011000 & \to & 001100 & \to & 000110 & \to & 100010 & \to & 110000. \end{array}$$

As described above, all one-to-one minimal threshold transformations we have found so far, including G in the above Example 5.3.1, are reflective through some Boolean isometries of order 2. However, there exists a compressible, minimal, one-to-one threshold transformation that is not reflective, as given in the following

Example 5.3.2. On the other hand, there exists an incompressible, non-minimal, one-to-one threshold transformation that is not reflective, as given in the following Example 5.3.3. Therefore, incompressibility as well as minimality is necessary for the conjecture.

Example 5.3.2 (Minimal, compressible, non-reflective one-to-one threshold transformation). $n = 6$. $F = [f_1, \ldots, f_6]$,

$$\begin{aligned} f_1 &= 1 \cdot \neg 2 \cdot \neg 3 \cdot \neg 4 \cdot (\neg 5 \vee \neg 6), \\ f_2 &= 1 \cdot 2 \cdot \neg 3 \cdot \neg 4 \cdot (\neg 5 \vee \neg 6), \\ f_3 &= f_4 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot (\neg 5 \vee \neg 6), \\ f_5 &= f_6 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6. \end{aligned}$$

F is compressible, since $3 \sim_F 4$. Suppose F is reflective through τJ^-. Since $|f_1| = |f_2| = |f_3| = |f_4| = 3$ and $|f_5| = |f_6| = 1$, $\tau\{5, 6\} = \{5, 6\}$ by Corollary 3.3.3 of Chapter 3. Since $f_6 \ni 111111 \to_F 111100$, $J = \{1, 2, 3, 4\}$ by Proposition 3.3.2 (i),(ii) of Chapter 3. However,

$$f_2 \ni 110000 \to_F 100000 \to_{J^-} 011100 \to_\tau x \notin f_{\tau 2},$$

which contradicts Proposition 3.3.2 (i) of Chapter 3.

Example 5.3.3 (Non-minimal, incompressible, non-reflective one-to-one threshold transformation). Let $G = \langle\langle g \rangle\rangle$, $g = p_1 \cdot p_2 \cdot p_3 \cdot p_4$, on $\mathbf{Q}^4$. Let $F = (3, 4)1^- G$. Then Then $F = [f_1, f_2, f_3, f_4]$,

$$\begin{aligned} f_1 &= 1 \cdot (\neg 2 \vee \neg 3 \vee \neg 4), \\ f_2 &= \neg 1 \cdot 2 \cdot 3 \cdot 4, \\ f_3 &= (\neg 1 \vee \neg 2) \cdot 3 \cdot \neg 4, \\ f_4 &= (\neg 1 \cdot \neg 2 \vee \neg 3) \cdot 4. \end{aligned}$$

Then $|f_1| = 7$, $|f_2| = 1$, $|f_3| = 3$, $|f_4| = 5$. Suppose that F is reflective through $T = \tau J^-$. Then $\tau = \iota$ by Corollary 3.3.3 of Chapter 3. On the other hand, $\{0000, 1111\}$ is the set of fixed points of F. Therefore, $J = \emptyset$ or $J = \mathbf{N}_4$. In either case, $F^2 = I$, which is not true. Therefore, F is not reflective through any Boolean isometry.

5.4 Compressible transformations

We gave circular, compressible, threshold transformations in Section 5.2. In this section we seek general, compressible minimal, threshold transformations constructed by orbit modification described in Section 5.1. First, we determine the cycle structures of threshold transformations obtained by expansion from the skew-circular transformation of Example 5.3.2.

Proposition 5.4.1 For any n and even k such that $k \leq 2n$, the cycle structure $\{(1, k), (2^n - k, 1)\}$ is realized by a threshold transformation.

Proof. Let

$$f_i = \begin{cases} x_1 \cdots x_i \cdot \neg x_{i+1} \cdots \neg x_n & \text{for } i \leq k/2 - 1, \\ x_1 x_n & \text{for } i \geq k/2. \end{cases}$$

Then $F = [f_1, ..., f_n]$ realizes the desired cycle structure. E.g. if $n = 5$, and $k = 6$, then

$$\begin{aligned} &11111 \to 11000 \to 10000 \to 00000 \\ &\quad \to 00111 \to 01111 \to 11111. \end{aligned}$$

□

The n-dimensional F of Proposition 5.4.1 is obtained from the nth face copies of the $(n-1)$-dimensional F by orbit connection. Note that $i \sim_F j$ for every $i, j \in \{k/2, k/2+1, .., n\}$.

Proposition 5.4.1 can be further generalized into the following proposition.

Proposition 5.4.2 If $n \geq 4$ is even, then the cycle structure $\{(1, n), (2, n-2), (2^n - 3n + 4, 1)\}$ is realized by a threshold transformation.

Proof. Let $n = 2m + 2$. Consider the transformation of Example 4.4.2 of Chapter 4 for $\mathbf{Q}^{2m}$. Specifically, $F = [f, \rho f, ..., \rho^{2m-1} f]$, where

$$f = p_1 \cdot p_2 \cdots p_m \cdot \neg p_{m+1} \cdots \neg p_{2m}.$$

From the $(2m+1, 2m+2)$th face copies of F through orbit connection, we obtain $G = [g_1, ..., g_{2m+2}]$, where

$$\begin{aligned} g_i &= (\rho^{i-1} f) \cdot (\neg p_{2m+1} \vee \neg p_{2m+2}) \text{ for } i = 1, ..., m, \\ g_i &= (\rho^{i-1} f) \cdot (p_{2m+1} \vee p_{2m+2}) \text{ for } i = m+1, ..., 2m, \\ g_{2m+1} &= g_{2m+2} = f \cdot p_{2m+1} \cdot p_{2m+2}. \end{aligned}$$

The threshold transformation G realizes the desired cycle structure. E.g.

$$\begin{aligned} &1000 \to 0100 \to 0111 \to 1011 \to 1000, \\ &1010 \to 0110 \to 1010, \\ &1001 \to 0101 \to 1001. \end{aligned}$$

□

Proposition 5.4.3 For any n, m and even k such that $1 \leq m < n$ and $k \leq 2m$, the cycle structure $\{(1, k+2), (2^{n-m} - 2, k), (2^n - k2^{n-m} + k - 2, 1)\}$ is realized by a threshold transformation.

Proof. Let the m-dimensional threshold transformation of Proposition 5.4.1 be $[f_1, ..., f_m]$. If $G = [g_1, ..., g_n]$ is defined as

$$g_i = \begin{cases} f_i(\neg p_{m+1} \vee ... \vee \neg p_n) & \text{for } i = 1, ..., m, \\ p_i p_n & \text{for } i = m+1, ..., n, \end{cases}$$

then G realizes the desired cycle structure. E.g. if $n = 5$, $m = 3$ and $k = 4$, then

$$\begin{aligned} &10000 \to 00000 \to 00011 \to 01111 \\ &\quad \to 11111 \to 11100 \to 10000, \\ &10001 \to 00001 \to 01101 \to 11101 \to 10001, \\ &10010 \to 00010 \to 01110 \to 11110 \to 10010, \\ &01110 \to 11110 \to 10010 \to 00010 \to 01110, \\ &01101 \to 11101 \to 10001 \to 00001 \to 01101. \end{aligned}$$

□

5.5 An enhanced Arimoto theorem

Arimoto's theorem (1963, Theorem 2, p. 20) proved that for any positive integer k such that $k \leq 2^n$, there exists a threshold transformation F of $\mathbf{Q}^n$ such that the graph of F has a k-cycle as its subgraph. Here we prove a stronger theorem using [ ]-representations of self-dual threshold transformations and construction methods introduced in Ueda (1992). Unlike Arimoto (1963), the present more combinatorial and simpler proof does not require the concept of *bordering points* and the construction of weight matrices and threshold vectors.

A special case of orbit modification is used in the proof of the next theorem. It was first introduced by Arimoto (1963).

Consider a self-dual transformation $F = [f_1, .., f_n]$ of $\mathbf{Q}^n$. Let q be a point of $\mathbf{Q}^n$. We modify f_i into g_i for each i in the following way. If $q_i = 1$, then let $g_i = f_i \cup q$ if $q \notin f_i$, and let $g_i = f_i \backslash q$ if $q \in f_i$. If $q_i = 0$ then let $g_i = f_i \cup \neg q$ if $\neg q \notin f_i$, and let $g_i = f_i \backslash \neg q$ if $\neg q \in f_i$. We have $Gq = \neg Fq$, $G(\neg q) = Fq$, and $Gx = Fx$ for every other x for $G = [g_1, .., g_n]$ thus constructed. Let C be the cycle or loop on which q is located, and let k be the length of C. If $\neg\mathbf{C} = \mathbf{C}$ then C is divided into two new $k/2$-cycles. If $\neg\mathbf{C} = \mathbf{D}$ for another k-cycle D, then C and D are united into one new $2k$-cycle. These situations are illustrated in Figs. 1 and 2. In particular, G is one-to one, if F is one-to-one. However, If F is a threshold transformation, G is not a threshold transformation except in special cases, since g_i are not necessarily threshold functions. The self-dual transformation G is called Arimoto's orbit modification of F at q.

$$
\begin{array}{ccccccccccccc}
 & & \circ & \to & \circ & \to & q & \to & r & \to & \circ & & \\
 & & & & \uparrow & & & & & & \downarrow & & \\
 & & & & \circ & \leftarrow & \neg r & \leftarrow & \neg q & \leftarrow & \circ & \leftarrow & \circ \\
 & & & & & & \Downarrow & & & & & & \\
\circ & \to & \circ & \to & q & & & & r & \to & \circ & & \\
 & & \uparrow & & \downarrow & & & & \uparrow & & \downarrow & & \\
 & & \circ & \leftarrow & \neg r & & & & \neg q & \leftarrow & \circ & \leftarrow & \circ
\end{array}
$$

Fig. 1

$$
\begin{array}{ccccccccccccc}
\circ & \to & \circ & \to & q & & & & r & \to & \circ & & \\
 & & \uparrow & & \downarrow & & & & \uparrow & & \downarrow & & \\
 & & \circ & \leftarrow & \neg r & & & & \neg q & \leftarrow & \circ & \leftarrow & \circ \\
 & & & & & & \Downarrow & & & & & & \\
 & & \circ & \to & \circ & \to & q & \to & r & \to & \circ & & \\
 & & & & \uparrow & & & & & & \downarrow & & \\
 & & & & \circ & \leftarrow & \neg r & \leftarrow & \neg q & \leftarrow & \circ & \leftarrow & \circ
\end{array}
$$

Fig. 2

Theorem 5.5.1 (Arimoto, 1963) There exist a threshold transformation F of $\mathbf{Q}^n$ such that GRAPH(F) consists of one 2^n-cycle, i.e. a Hamiltonian cycle.

Proof. We construct a desired F by iterating copying and Arimoto's orbit modification. (1) For $n = 1$, one 2-cycle is realized by the transformation $F = [f_1]$, $f_1 = \{1\} = p_1$. GRAPH(F) is

$$1 \to 0 \to 1.$$

(2) For $n = 2$, two 2-cycles are realized by the face copies $F = [f_1, f_2]$, $f_1 = p_1$, $f_2 = \emptyset$. GRAPH(F) is

$$\begin{array}{c} 11 \to 01 \to 11, \\ 00 \to 10 \to 00. \end{array}$$

(3) For $n = 2$, one 2^2-cycle is realized by $F = [f_1, f_2]$, which is Arimoto's orbit modification at 11,

$$\begin{array}{rcl} f_1 &=& p_1 \backslash 11 = p_1 \cdot \neg p_2, \\ f_2 &=& \{11\} = p_1 \cdot p_2, \end{array}$$

where $\backslash 11$ means the deletion of 11. GRAPH(F) is now

$$\begin{array}{ccccc} 11 & & 01 & \to & 11. \\ \downarrow & & \uparrow & & \\ 10 & \to & 00 & & \end{array}$$

(4) For $n = 3$, two 2^2-cycles are realized by $F = [f_1, f_2, f_3]$,

$$\begin{array}{rcl} f_1 &=& p_1 \cdot \neg p_2, \\ f_2 &=& p_1 \cdot p_2, \\ f_3 &=& \emptyset. \end{array}$$

GRAPH(F) is

$$\begin{array}{c} 111 \to 101 \to 001 \to 011 \to 111, \\ 000 \to 010 \to 110 \to 100 \to 000. \end{array}$$

(5) For $n = 3$, one 2^3-cycle is realized by $F = [f_1, f_2, f_3]$,

$$\begin{array}{rcl} f_1 &=& (p_1 \cdot \neg p_2) \cup 111 = p_1 \cdot (\neg p_2 \vee p_3), \\ f_2 &=& (p_1 \cdot p_2) \backslash 111 = p_1 \cdot p_2 \cdot \neg p_3, \\ f_3 &=& \{111\} = p_1 \cdot p_2 \cdot p_3, \end{array}$$

where $\cup 111$ means the addition of 111. GRAPH(F) is

$$\begin{array}{ccccccccccccc} 111 & & & & & & 101 & \to & 001 & \to & 011 & \to & 111. \\ \downarrow & & & & & & \uparrow & & & & & & \\ 010 & \to & 110 & \to & 100 & \to & 000 & & & & & & \end{array}$$

In general, if one 2^k-cycle

$$1^k = x^{(1)} \to x^{(2)} \to ... \to x^{(2^k)} \to x^{(1)}$$

is realized by $F = [f_1, .., f_k]$, then two 2^k-cycles

$$\begin{array}{ccccccccc} 1^{k+1} = x^{(1)}1 & \to & x^{(2)}1 & \to & ... & \to & x^{(2^k)} & \to & x^{(1)}1, \\ 0^{k+1} = (\mathbf{N}_k^- x^{(1)})0 & \to & (\mathbf{N}_k^- x^{(2)})0 & \to & ... & \to & (\mathbf{N}_k^- x^{(2^k)})0 & \to & (\mathbf{N}_k^- x^{(1)})0 \end{array}$$

are realized by $F\prime = [f_1 \circ P_{\mathbf{N}_k}, .., f_k \circ P_{\mathbf{N}_k}, \emptyset]$. Here $1^k = 1 \cdots 1 \in \mathbf{Q}^k$ and $0^{k+1} = 0 \cdots 0 \in \mathbf{Q}^{k+1}$. Therefore, one 2^{k+1}-cycle

$$\begin{array}{ccccccccccccc} x^{(1)}1 & & & & & & & x^{(2)}1 & \to & x^{(3)}1 & \to \ \ldots \ \to & x^{(2^k)}1 & \to \ x^{(1)}1, \\ \downarrow & & & & & & & & \nwarrow & & & & \\ (\neg x^{(2)})0 & \to & (\neg x^{(3)})0 & \to & \ldots & \to & (\neg x^{(2^k)})0 & \to & (\neg x^{(1)})0 & & & & \end{array}$$

is realized by a self-dual transformation G such that $G(x^{(1)}1) = \neg F\prime(x^{(1)}1)$ and $G(x^{(j)}1) = F\prime(x^{(j)}1)$ for every $j \neq 1$. This $G = [g_1, .., g_{k+1}]$ can be defined by

$$g_i = \begin{cases} f_i \circ P_{\mathbf{N}_k} \backslash 1^{k+1} & \text{if } 1^{k+1} \in f_i \circ P_{\mathbf{N}_k}, \\ f_i \circ P_{\mathbf{N}_k} \cup 1^{k+1} & \text{if } 1^{k+1} \notin f_i \circ P_{\mathbf{N}_k}, \\ g_{k+1} = \{1^{k+1}\} & . \end{cases}$$

Further, if $1^k \in f_i$ then $1^{k+1} \in f_i \circ P_{\mathbf{N}_k}$; if $1^k \notin f_i$ then $1^{k+1} \notin f_i \circ P_{\mathbf{N}_k}$. Therefore, if $1^k \in f_k$, $1^k \notin f_{k-1}$, $1^k \in f_{k-2}$, and $1^k \notin f_{k-3}$,..,for F, then $g_k = f_k \circ P_{\mathbf{N}_k} \backslash 1^{k+1}$, $g_{k-1} = f_{k-1} \circ P_{\mathbf{N}_k} \cup 1^{k+1}$, $g_{k-2} = f_{k-2} \circ P_{\mathbf{N}_k} \backslash 1^{k+1}$, $g_{k-3} = f_{k-3} \circ P_{\mathbf{N}_k} \cup 1^{k+1}$, ...

By repeating the above process of constructing G from F, we obtain $F = [f_1, .., f_n]$,

$$f_1 = \begin{cases} p_1 \cdot (\neg p_2 \vee p_3 \cdot (\neg p_4 \vee p_5 \cdot (\neg p_6 \vee p_7 \cdot (.. \cdot (\neg p_{n-4} \vee p_{n-3} \cdot (\neg p_{n-2} \vee p_{n-1} \cdot \neg p_n))..) \\ \quad \text{if } n \text{ is even}, \\ p_1 \cdot (\neg p_2 \vee p_3 \cdot (\neg p_4 \vee p_5 \cdot (\neg p_6 \vee .. \vee p_{n-4} \cdot (\neg p_{n-3} \vee p_{n-2} \cdot (\neg p_{n-1} \vee p_n))..) \\ \quad \text{if } n \text{ is odd}, \end{cases}$$

....

$$\begin{aligned} f_{n-3} &= p_1 \cdots p_{n-3} \cdot (\neg p_{n-2} \vee p_{n-1} \cdot \neg p_n), \\ f_{n-2} &= p_1 \cdots p_{n-2} \cdot (\neg p_{n-1} \vee p_n), \\ f_{n-1} &= p_1 \cdots p_{n-1} \cdot \neg p_n, \\ f_n &= p_1 \cdots p_n. \end{aligned}$$

GRAPH(F) consists of one 2^n-cycle, and f_i is a threshold function for every i by Proposition 4.1.2 of Chapter 4 (Read these equations upward from the bottom and backward from the right). $\square$

The next tables illustrate the process of creating F for $n = 3$ in the proof of Theorem 5.5.1. In the tables, each left column shows $f_1, .., f_n$, and each right column shows $\neg f_1, \neg f_2, .., \neg f_n$, where a function f is expressed by $f^{-1}1$, and * denotes $\{0, 1\}$.

$$1 \mid 0 \quad \to \quad \begin{array}{c|c} 1* & 0* \\ \hline \emptyset & \emptyset \end{array} \quad \to \quad \begin{array}{c|c} 1*\backslash 11 & 0*\backslash 00 \\ \hline 11 & 00 \end{array} = \begin{array}{c|c} 10 & 01 \\ \hline 11 & 00 \end{array}$$

$$\to \quad \begin{array}{c|c} 10* & 01* \\ \hline 11* & 00* \\ \hline \emptyset & \emptyset \end{array} \quad \to \quad \begin{array}{c|c} 10*\cup 111 & 01*\cup 000 \\ \hline 11*\backslash 111 & 00*\backslash 000 \\ \hline 111 & 000 \end{array} = \begin{array}{c|c} 10* & 01* \\ 111 & 000 \\ \hline 110 & 001 \\ \hline 111 & 000 \end{array}$$

Theorem 5.5.2 (Ueda, 1997) There exists a threshold transformation H of $\mathbf{Q}^n$ such that for any $k \leq 2^n$, GRAPH(H) has a k-cycle that is the only cycle or loop.

Proof. Consider the one-to-one self-dual threshold transformation $F = [f_1, .., f_n]$ of $\mathbf{Q}^n$ constructed in the proof of Theorem 5.5.1 such that GRAPH(F) consists of one 2^n-cycle, which is

$$1^n = x^{(1)} \to x^{(2)} \to ... \to x^{(2^n)} \to x^{(1)}.$$

Let $F\prime$ be the transformation of $\mathbf{Q}^{n+1}$ defined by $F\prime = [f_1 \circ P_{\mathbf{N}}, .., f_n \circ P_{\mathbf{N}}, \emptyset]$. Then

$$\begin{array}{ccccccccccc} 1^{n+1} & = & x^{(1)}1 & \to & x^{(2)}1 & \to & ... & \to & x^{(2^n)}1 & \to & x^{(1)}1, \\ 0^{n+1} & = & (\neg x^{(1)})0 & \to & (\neg x^{(2)})0 & \to & ... & \to & (\neg x^{(2^n)})0 & \to & (\neg x^1)0 \end{array}$$

are the two 2^n-cycles of $F\prime$. The modification $x^{(1)}1 \to x^{(j)}1$ and $(\neg x^{(1)})0 \to (\neg x^{(j)})0$ changes GRAPH($F\prime$) into

$$\begin{array}{ccccccccccccc}
x^{(2)}1 & \to & \dots & \to & x^{(j-1)}1 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & x^{(1)}1 & \to & x^{(j)}1 & \to & \dots & \to & x^{(2^n)}1 & \to & x^{(1)}1, & & \\
(\neg x^{(2)})0 & \to & \dots & \to & (\neg x^{(j-1)})0 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & (\neg x^{(1)})0 & \to & (\neg x^{(j)})0 & \to & \dots & \to & (\neg x^{(2^n)})0 & \to & (\neg x^{(1)})0, & &
\end{array}$$

which has two $(2^n - j + 2)$-cycles that are the only cycles or loops. $F\prime$ has been changed by this modification into $G = [g_1, .., g_n, \emptyset]$, where

$$g_i = \begin{cases} f\prime_I & \text{if } x_i^{(2)} = x_i^{(j)}, \\ f\prime_I \cup 1^{n+1} & \text{if } x_i^{(2)} = 1\text{and } x_i^{(j)} = 0, \\ f\prime_i \backslash 1^{n+1} & \text{if } x_i^{(2)} = 0 \text{ and } x_i^{(j)} = 1. \end{cases}$$

As shown in the proof of Theorem 5.5.1, g_i is a threshold function for every i. Let the transformation H of $\mathbf{Q}^n$ be defined by

$$H_i(x_1, .., x_n) = G_i(x_1, .., x_n, 1)$$

for $i = 1, .., n$. Then H is a threshold transformation by Proposition 4.3.6 of Chapter 4. Further, GRAPH(H) is

$$\begin{array}{ccccccccccc}
x^{(2)} & \to & \dots & \to & x^{(j-1)} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & x^{(1)} & \to & x^{(j)} & \to & \dots & \to & x^{(2^n)} & \to & x^{(1)},
\end{array}$$

which has one $(2^n - j + 2)$-cycle, and this cycle is the only cycle or loop. $\square$

The next tables illustrate the process of creating H for $n = 3$ having one 5-cycle in the proof of Theorem 5.5.2.

$$F\prime : \begin{array}{c|c} 10** & 01** \\ 111* & 000* \\ \hline 110* & 001* \\ \hline 111* & 000* \\ \hline \emptyset & \emptyset \end{array} \quad \to \quad G : \begin{array}{c|c} 10** & 01** \\ 111* & 000* \\ \hline 110* & 001* \\ 1111 & 0000 \\ \hline 111* & 000* \\ \hline \emptyset & \emptyset \end{array}$$

GRAPH(G) is

$$\begin{array}{ccccccccccccc}
0101 & \to & 1101 & \to & 1001 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & 1111 & \to & 0001 & \to & 1011 & \to & 0011 & \to & 0111 & \to & 1111, \\
1010 & \to & 0010 & \to & 0110 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & 0000 & \to & 1110 & \to & 0100 & \to & 1100 & \to & 1000 & \to & 0000.
\end{array}$$

GRAPH(H) is

$$\begin{array}{ccccccccccccc}
010 & \to & 110 & \to & 100 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & 111 & \to & 000 & \to & 101 & \to & 001 & \to & 011 & \to & 111.
\end{array}$$